MA114 Summer 2018
Worksheet 15 – Taylor Series – 7/09/18

1. Let $f(x) = 1 + 2x + 3x^2 + 4x^3$. Find the Taylor series for $f(x)$ at $x = 0$ (the Maclaurin series).

$f(x) = 1 + 2x + 3x^2 + 4x^3$ $\quad f(0) = 1$

$f'(x) = 2 + 6x + 12x^2$ $\quad f'(0) = 2$

$f''(x) = 6 + 24x$ $\quad f''(0) = 6$

$f'''(x) = 24$ $\quad f'''(0) = 24$

$f^{(4)}(x) = 0$ $\quad f^{(4)}(0) = 0$

$f^{(n)}(0) = 0$ for $n \geq 4$

So the Taylor series is

$$\frac{1}{0!} + \frac{2}{1!}x + \frac{6}{2!}x^2 + \frac{24}{3!}x^3 + 0 \ldots$$
$$= 1 + 2x + 3x^2 + 4x^3$$
$$= f(x).$$

2. Find the Taylor series expansion about $x = 1$ of $\sin(\pi x)$.

$f(x) = \sin(\pi x)$ $\quad f(1) = 0$

$f'(x) = \pi\cos(\pi x)$ $\quad f'(1) = -\pi$

$f''(x) = -\pi^2\sin(\pi x)$ $\quad f''(1) = 0$

$f'''(x) = -\pi^3\cos(\pi x)$ $\quad f'''(1) = \pi^3$

$f^{(4)}(x) = \pi^4\sin(\pi x)$ $\quad f^{(4)}(1) = 0$

$f^{(5)}(x) = \pi^5\cos(\pi x)$ $\quad f^{(5)}(1) = -\pi^5$

So the Taylor series is

$$0 - \frac{\pi}{1!}(x-1) + 0(x-1)^2 + \frac{\pi^3}{3!}(x-1)^3 + 0(x-1)^4 - \frac{\pi^5}{5!}(x-1)^5 + \ldots$$
$$= \boxed{\sum_{n=0}^{\infty} \frac{(-1)^{n+1}\pi^{2n+1}}{(2n+1)!}(x-1)^{2n+1}}$$

3. Use known Maclaurin series to find the Maclaurin expansions of

(a) $f(x) = xe^{2x}$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$f(x) = xe^{2x} = x\sum_{n=0}^{\infty}\frac{(2x)^n}{n!} = \sum_{n=0}^{\infty}\frac{x2^nx^n}{n!} = \boxed{\sum_{n=0}^{\infty}\frac{2^nx^{n+1}}{n!}} = \boxed{\sum_{n=1}^{\infty}\frac{2^{n-1}x^n}{(n-1)!}}$$

(b) $g(y) = 2\cosh(y) = e^y + e^{-y}$.

$$g(y) = \sum_{n=0}^{\infty}\frac{y^n}{n!} + \sum_{n=0}^{\infty}\frac{(-y)^n}{n!} = \sum_{n=0}^{\infty}\frac{y^n + (-1)^ny^n}{n!} = \sum_{n=0}^{\infty}\frac{(1+(-1)^n)y^n}{n!}$$

$(1 + (-1)^n) = 2$ if n is even and 0 if n is odd

$$= \sum_{n=0}^{\infty}\frac{2y^{2n}}{(2n)!}$$
$$= 2 + \frac{2y^2}{2!} + \frac{2y^4}{4!} + \frac{2y^6}{6!} + \ldots$$

(c) $h(t) = t^5 \sin(3t^2)$.

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$h(t) = t^5 \sin(3t^2) = t^5 \sum_{n=0}^{\infty} \frac{(-1)^n (3t^2)^{2n+1}}{(2n+1)!} = t^5 \sum_{n=0}^{\infty} \frac{(-1)^n 3^{2n+1} t^{4n+2}}{(2n+1)!}$$

$$= \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n 3^{2n+1} t^{4n+7}}{(2n+1)!}}$$

4. Approximate the following integral using a 6th order Taylor polynomial for $\cos(x)$:

$$\int_0^1 x\cos(x^3)\,dx.$$

For $\cos(x)$, $T_6(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720}$.

So $\int_0^1 x\cos(x^3)dx \approx \int_0^1 x\left(1 - \frac{x^6}{2} + \frac{x^{12}}{24} - \frac{x^{18}}{720}\right)dx$

$$= \int_0^1 x - \frac{x^7}{2} + \frac{x^{13}}{24} - \frac{x^{19}}{720}\,dx$$

$$= \frac{1}{2}x^2 - \frac{1}{16}x^8 + \frac{1}{14\cdot 24}x^{14} - \frac{1}{20\cdot 720}x^{20}\Big|_0^1$$

$$= \frac{1}{2} - \frac{1}{16} + \frac{1}{14\cdot 24} - \frac{1}{20\cdot 720}$$

$$= 0.4404067$$